Module 2: Transformations and Scene Creation

Linear Algebra Review Sheet Solutions

1. Let $a = (1, 2, 3)$, $b = (-1, 4, 0)$, $c = (3, -2, 4)$. Compute the following:
   a. $a \bullet a = 1\times 1 + 2\times 2 + 3\times 3 = 1 + 4 + 9 = 14$
   b. $\|a\|^2 = 1^2 + 2^2 + 3^2 = 14$
   c. $a \bullet b = 1\times(-1) + 2\times 4 + 3\times 0 = -1 + 8 + 0 = 7$
   d. $b \bullet a = (-1)\times 1 + 4\times 2 + 0\times 3 = -1 + 8 + 0 = 7$
   e. $$\begin{aligned} a \bullet (b + c) &= (1, 2, 3) \bullet ((-1, 4, 0) + (3, -2, 4)) \\ &= (1, 2, 3) \bullet (2, 2, 4) = 1\times 2 + 2\times 2 + 3\times 4 = 18 \end{aligned}$$
   f. $a \bullet b + a \bullet c = 7 + (1, 2, 3) \bullet (3, -2, 4) = 7 + 3 - 4 + 12 = 18$

2. Find $2u + 5(v + w)$ where $u = (1, 7, -2)$, $v = (3, -1, 0)$, $w = (-2, 2, -2)$.

Answer:

$$\begin{aligned} 2u + 5(v + w) &= 2(1,7,-2) + 5((3,-1,0) + (-2,2,-2)) \\ &= (2,14,-4) + 5(1,1,-2) = (7,19,-14). \end{aligned}$$

3. Construct the vector with unit length given by $\frac{v}{\|v\|}$, where $v = (2, -4, 3)$.

Answer:

$$\|v\| = \sqrt{4 + 16 + 9} = \sqrt{29}$$

$$\frac{v}{\|v\|} = \frac{1}{\sqrt{29}}(2,-4,3) = \left(\frac{2}{\sqrt{29}}, -\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}\right).$$

4. Calculate $u \cdot (v + w)$, where $u = (1, 2, 2)$, $v = (-3, 0, -2)$, $w = (-1, -1, 4)$.

Answer:

$$u \cdot (v + w) = (1,2,2) \cdot ((-3,0,-2) + (-1,-1,4)) = (1,2,2) \cdot (-4,-1,2) = -4 - 2 + 4 = -2.$$

5. Find the angle between the vectors $a = (2, 2, -1)$ and $b = (5, -4, 2)$.

Answer:

$$\cos\theta = \frac{a \bullet b}{\|a\|\,\|b\|} = \frac{10 - 8 - 2}{\sqrt{9}\,\sqrt{38}} = 0 \quad \Rightarrow \theta = 90.$$

6. Find two unit vectors orthogonal to both $\left(1, 1, \frac{1}{2}\right)$ and $\left(2, 0, 1\right)$.

Answer: We need $u \bullet (1, 1, \frac{1}{2}) = 0, \quad u \bullet (2, 0, 1) = 0,$ *and the same for a vector v. Working with vector u,* $u_1 + u_2 + \frac{1}{2}u_3 = 0, \quad 2u_1 + u_3 = 0.$ *So* $u_2=0$*. Choose* $u = (1, 0, -2), \quad v = (-\frac{1}{2}, 0, 1).$ *Now normalise to get*

$$\frac{u}{\|u\|} = \frac{1}{\sqrt{5}}\left(1, 0, -2\right) = (\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}),$$

$$\frac{v}{\|v\|} = \frac{1}{\frac{\sqrt{5}}{2}}\left(-\frac{1}{2}, 0, 1\right) = (-\frac{1}{2}\times\frac{2}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}).$$

7. Find the projection of w onto v, where $w = (2,3,1), v = (3,0,0)$, as well as the associated orthogonal vector u.

Answer:

$$\|w\| = \sqrt{4+9+1} = \sqrt{14}, \quad \|v\| = 3$$

$$\alpha v = \frac{w \cdot v}{v \cdot v} = \frac{(2,3,1)\cdot(3,0,0)}{9} = \frac{2}{3}$$

$$\textit{projection} = \alpha v = \frac{2}{3}(3,0,0) = (2,0,0).$$

$$\textit{associated orthogonal vector} \quad u = w - \frac{w \cdot v}{v \cdot v} v = (2,3,1) - (2,0,0) = (0,3,1)$$

8. What is the vector that is perpendicular to both $u = (1,2,3)$ and $v = (-1,4,-2)$? Use the cross-product operator.

Answer:

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 4 & -2 \end{vmatrix} = -16i - j + 6k, \quad u \times v = (-16,-1,6).$$

9. What is the area of the parallelogram with sides u and v for $u = (1,2,3)$ and $v = (-1,4,-2)$? You need to compute $\|u \times v\|$.

Answer:

$$\|u \times v\| = \|(-16,-1,6)\| = \sqrt{16^2 + 1^2 + 6^2} = \sqrt{293} \approx 17.117$$

10. For $u = (1, 2, 0)$ and $v = (3, a, 1)$, what value of a is required for u and v to be perpendicular?

Answer:

$$u \cdot v = (1,2,0) \cdot (3,a,1) = 3 + 2a = 0 \Rightarrow 3 = -2a \Rightarrow a = -\frac{3}{2}.$$

11. Calculate the unit normal to the plane given by $2x - 3y + 4z = 5$.

Answer:

$$\textit{normal} \quad n = (2,-3,4), \quad \|n\| = \sqrt{4+9+16} = \sqrt{29}$$

$$\textit{unit normal} = \left(\frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)$$

12. Find the normal to the plane that passes through the points $A = (1, 2, 1), \quad B = (-3, -1, 2), \quad C = (1, -1, -1)$. Hint: find the cross-product $(B - A) \times (C - A)$.

Answer:

$$B - A = (-3-1, -1-2, 2-1) = (-4, -3, 1)$$

$$C - A = (1-1, -1-2, -1-1) = (0, -3, -2)$$

$$(B - A) \times (C - A) = \begin{vmatrix} i & j & k \\ -4 & -3 & 1 \\ 0 & -3 & -2 \end{vmatrix} = i(9) - j(8) + k(12).$$

So any non-zero scalar multiple of the vector (9, -8, 12) is perpendicular to this plane.

13. Find the equation of the plane in the preceding question. Hint: use the equation $n \bullet (P - P_0) = 0$ where $P = (x, y, z)$ and P_0 is one of the specified points.

Answer:

$$\begin{aligned} n \bullet (P - P_0) &= (9, -8, 12) \bullet ((x, y, z) - (1, 2, 1)) \quad \textit{with } P_0 = A \\ &= (9, -8, 12) \bullet (x - 1, y - 2, z - 1) \\ &= 9 \times (x - 1) - 8 \times (y - 2) + 12 \times (z - 1) \\ &= 9x - 8y + 12z - 9 + 16 - 12 \\ &= 9x - 8y + 12z - 5 \end{aligned}$$

so the equation of the plane is $9x - 8y + 12z = 5$.

14. Two planes are parallel if their normal vectors are parallel. If they are not parallel, they meet in a straight line and the angle between them is the acute angle between their normal vectors. If the two normal vectors are orthogonal, the planes are perpendicular. Are the following planes parallel, perpendicular or neither? If neither, compute the angle between them (using $\cos\theta = \dfrac{n_1 \bullet n_2}{\|n_1\|\,\|n_2\|}$).

a. $x + y + z = 3, \quad 2x - y - z = 2$

Answer:

$$n_1 = (1, 1, 1), \quad n_2 = (2, -1, -1) \quad (\textit{so they are not parallel})$$
$$n_1 \bullet n_2 = 2 - 1 - 1 = 0 \quad \textit{so they are perpendicular}.$$

b. $2x + y - z = 4, \quad 2x - 3y - 2z = 2$.

Answer:

$$n_1 = (2, 1, -1), \quad n_2 = (2, -3, -2) \quad (\textit{not parallel})$$
$$n_1 \bullet n_2 = 4 - 3 + 2 = 3 \qquad (\textit{not perpendicular})$$
$$\cos\theta = \frac{n_1 \bullet n_2}{\|n_1\|\,\|n_2\|} = \frac{3}{\sqrt{6}\,\sqrt{17}} = \frac{3}{\sqrt{102}}$$
$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{102}}\right) \approx 72.7^\circ.$$

15. Calculate the following matrix products:

Answer:

$$(a) \quad \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 & 2 \\ 1 & 4 & -3 \\ 5 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 3 & 4 \\ 10 & 9 & -10 \\ -3 & 10 & 1 \end{pmatrix}$$

$$(b) \quad \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 2 & -1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 7 & 4 \\ 1 & 3 & 3 & -3 \\ 1 & -2 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

16. Which matrix (M_1, M_2, M_3) is the inverse of $M = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}$?

$$M_1 = \begin{pmatrix} \frac{12}{25} & \frac{1}{50} & -\frac{3}{25} \\ -\frac{4}{25} & \frac{4}{25} & \frac{1}{25} \\ \frac{1}{25} & -\frac{1}{25} & \frac{6}{25} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \frac{12}{25} & \frac{1}{25} & -\frac{3}{25} \\ -\frac{4}{25} & \frac{8}{25} & \frac{1}{25} \\ \frac{1}{25} & -\frac{2}{25} & \frac{6}{25} \end{pmatrix}, \quad or \quad M_3 = \begin{pmatrix} \frac{12}{25} & \frac{1}{25} & -\frac{3}{25} \\ -\frac{4}{25} & \frac{8}{25} & \frac{1}{25} \\ \frac{1}{25} & -\frac{2}{25} & \frac{4}{25} \end{pmatrix}.$$

Answer: M_2

17. Demonstrate that a rotation followed by a translation is not the same as a translation followed by a rotation, for the particular example given by a rotation of 30 degrees around the *z*-axis and a translation of 1, 2 and –1 in the *x*, *y* and *z*-directions respectively. In other words, form the *R* and *T* matrices, and compute *RT* and *TR*.

Answer:

$$R = \begin{pmatrix} \cos 30 & -\sin 30 & 0 & 0 \\ \sin 30 & \cos 30 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R\,T = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} + \sqrt{3} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T\,R = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$