COMP3201 – Computer Graphics

2. Module 2: Transformations and Scene Creation

2.1 Linear Algebra Review

2.1.1 Introduction

Computer Graphics uses various geometric objects, each of which has a shape, a position and an orientation. The geometric objects are constructed from a set of points, which have been specified in accordance with some coordinate system.
This section of the notes discusses the geometric representation of points and vectors, and gives the basic tools for manipulating vectors and matrices. These tools are used when applying transformations to geometric/graphics objects as well as for calculating normals which are used in lighting effects.

2.1.2 Terminology

A point is a location in space, and is defined by specifying its coordinates (e.g. a point in 3D space could be (1, 2, 3)).

A vector has a length (or magnitude) and a direction, but no location. It is sometimes refered to as a “directed line segment”. Given two points A and B, we can construct a directed line segment that connects them. For example, in 2D (it’s easier to draw), let A be the point (1,1) and B the point (5,4); v is then an example of a vector of length 5 connecting A to B; it can be written $v = (4, 3)$ meaning that, from its starting point, it represents a movement 4 units horizontally for every 3 units vertically.

Given two points A and B, we can form the parametric representation of the line passing through these points. It is given by

$$p(t) = \overline{A} + t\,(\overline{B} - \overline{A}),$$

where $\overline{A}$, $\overline{B}$ are the vectors from the origin to the points A and B, respectively. All points on the line segment $\overline{AB}$ are given by $p(t)$, for $t \in [0,1]$. For $-\infty < t < \infty$, $p(t)$ defines the infinite line through these points.

2.1.3 Vector Arithmetic

New vectors can be formed by adding two (or more) vectors (component-wise) or through multiplication (of each component) by a scalar which in this context is a real number $\alpha \in R$. Geometrically, multiplication by a scalar has the effect of changing the length of the vector or (if the scalar is negative) changing its length and

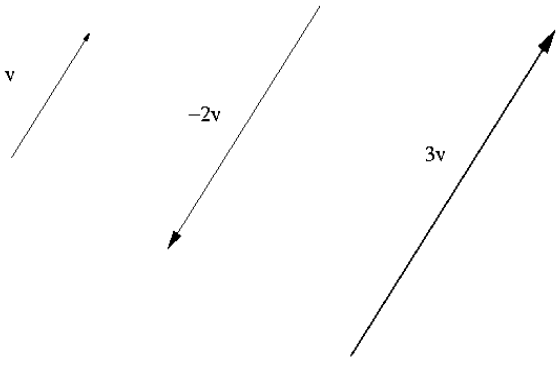

Figure 2.1

direction. For example, Figure 2.1 shows the results of defining $v = (2, 3)$; then $3v = (6, 9)$, while $-2v = (-4, -6)$.

There are two ways to visualise the addition of two vectors; one way is to use the parallelogram law (as used with forces):

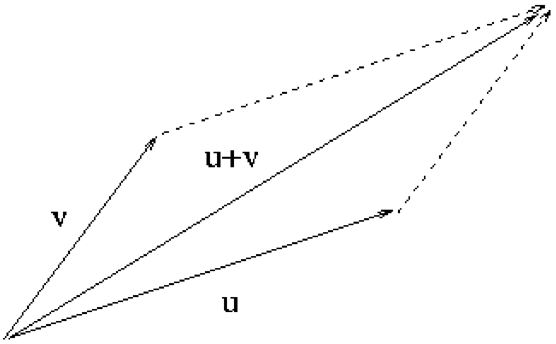

Figure 2.2

Another way is to use the "tail-to-head" displacements rule, where vector *v* is started at the top of *u* so that *u*+*v* goes from the start of *u* to the tip of *v*:

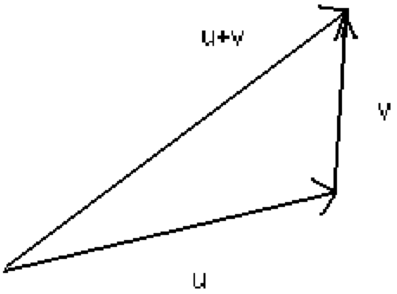

Figure 2.3

Of course, both pictorial representations have the same result for *u*+*v*.

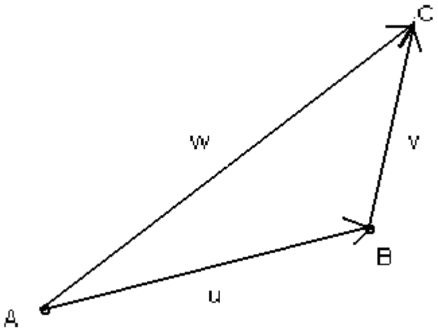

Figure 2.4

We can also use points to describe this geometric representation. In the above diagram, *u* is the vector joining point *A* to point *B*; that is, $B = A + u$ or $u = B - A$; similarly, $v = C - B$, and $w = C - A = C - B + B - A = v + u$.

Subtraction of a vector is just the addition of a vector with a vector that has been scaled by –1: $w = u - v = u + (-1)v$.

A linear combination of vectors is the sum of individual vectors that have been multiplied by scalars; for example, a linear combination of n vectors $v_1, \cdots, v_n$ is $w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, where $\alpha_i \in R, i = 1, \cdots, n$.

2.1.4 Vector Arithmetic Rules

The vectors in Computer Graphics can be considered as directed line segments and belong to Euclidean space, which provides a set of rules for the addition of vectors and multiplication by a scalar as well as a measure of distance.

The following rules hold, for $a, b \in R$, and vectors $u, v, w \in R^n$.

(i)	associativity	$(u + v) + w = u + (v + w)$
(ii)	commutativity	$u + v = v + u$
(iii)	zero identity	$0 + v = v + 0 = v$
(iv)	additive inverse	$u + (-u) = (-u) + u = 0$
(v)	multiplication	$(ab)u = a(bu)$
(vi)	scalar distributive	$(a + b)u = au + bu$
(vii)	vector distributive	$a(u + v) = au + av$.

2.1.5 Magnitude of a Vector

The magnitude of a vector v is its length; it is denoted $\|v\|$, and is computed using the Pythagoras rule

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2} \geq 0.$$

For example, if $v = (4, 3)$, then $\|v\| = \sqrt{16 + 9} = \sqrt{25} = 5$.

Often, vectors need to be scaled to have unit length (that is, length equal to one), and this is achieved by scaling v by $1/\|v\|$.

Note that

- $\|v\| \geq 0$
- $\|v\| = 0$ if and only if $v = 0$
- $\|av\| = |a| \, \|v\|$ where $|a|$ is the absolute value of the real number a.
- $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality)

2.1.6 Dot Products and Orthogonality

The dot product (also sometimes called scalar product or inner product) of two vectors u and v is a scalar value, and is written $u \cdot v$. If $u = (u_1, \cdots, u_n)$ and $v = (v_1, \cdots, v_n)$, then $u \cdot v = u_1 v_1 + \cdots + u_n v_n$. For example, if $u = (1, 2)$ and $v = (3, 4)$, then $u \cdot v = 1 \times 3 + 2 \times 4 = 11$.

Also, $\|u\|^2 = u \cdot u$ gives the relationship between magnitude and dot product.

Another relationship is $u \cdot v = \|u\|\,\|v\| \cos\alpha$, where α is the angle between the two vectors. In particular, if two vectors *u* and *v* are orthogonal (that is, they meet at right angles), then $u \cdot v = 0$.

Some properties of the dot product are

- $u \cdot v = v \cdot u$
- $(\alpha u) \cdot v = \alpha (u \cdot v)$
- $u \cdot (v + w) = u \cdot v + u \cdot w$
- $u \cdot u = \|u\|^2$
- $|u \cdot v| \le \|u\|\,\|v\|$.

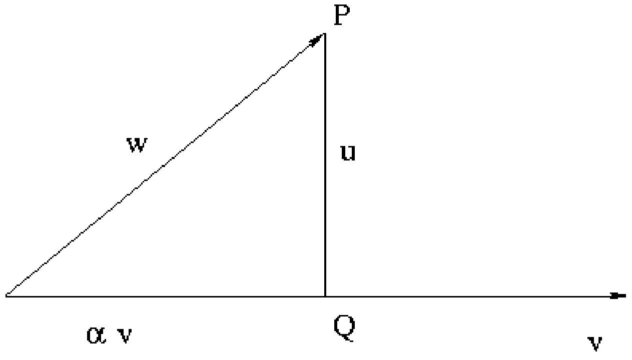

Figure 2.5

The property of orthogonality is used to find the projection of a point to a line. For example, to find the projection of *w* onto *v*, consider the vector *u*, orthogonal to *v*. From Figure 2.5, we can see that $w = \alpha v + u$, and we also know that $u \cdot v = 0$. So

$$w \cdot v = \alpha v \cdot v + u \cdot v = \alpha v \cdot v \quad \Rightarrow \quad \alpha = \frac{w \cdot v}{v \cdot v}.$$

So the projection of *w* onto *v* is αv, and the orthogonal vector *u* is

$$u = w - \frac{w \cdot v}{v \cdot v} v.$$

2.1.7 <u>Cross product</u>

The cross product of two 3D vectors (sometimes called the vector product) results in a new vector that is perpendicular to each of the two vectors being multiplied together. It is used for calculating the surface normal at a particular point.

It is denoted $w = u \times v$, and by perpendicularity, $w \cdot u = 0, \quad w \cdot v = 0$ (which is also written $(u \times v) \cdot u = 0, \quad (u \times v) \cdot v = 0$).

For α the angle between *u* and *v*, $\|w\| = \|u \times v\| = \|u\|\,\|v\| \sin\alpha$. In other words, the magnitude of the cross product of two vectors is equal to the area of the parallelogram whose sides are $\|u\|$ and $\|v\|$.

So, assuming *u* and *v* are non-zero, then $u \times v$ is zero if and only if $\sin\alpha = 0$, which means that *u* and *v* are parallel.

If $u = (u_1, u_2, u_3)^T$ and $v = (v_1, v_2, v_3)^T$, then

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = u \times v = \begin{pmatrix} u_2 v_3 - v_2 u_3 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \equiv \det \begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$

See later for the definition of det, the determinant of a matrix.

Some properties of the cross product are:

- $u \times v = -v \times u$
- $u \times (v + w) = u \times v + u \times w$
- For the standard unit vectors (*i*=(1, 0, 0), *j*=(0, 1, 0), *k*=(0, 0, 1)),
  $i \times j = k, \quad j \times k = i, \quad k \times i = j$

2.1.8 Matrices

A matrix *M* of dimension *m* x *n* has *m* rows and *n* columns. It is said to be a square matrix if *m* = *n*. The elements of *M* are m_{ij} where *i* indicates the row number and *j* the column number. For example, the 2 x 3 matrix *M* is $\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix}$.

The multiplication of *M* by a scalar α is calculated by multiplying each component of *M* by α:

$$\alpha M = \begin{pmatrix} \alpha m_{11} & \alpha m_{12} & \alpha m_{13} \\ \alpha m_{21} & \alpha m_{22} & \alpha m_{23} \end{pmatrix}.$$

Addition of matrices is straightforward, and computed component-wise; the matrices to be added must have the same dimension (that is, the same shape):

$$M + N = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \vdots \end{pmatrix} + \begin{pmatrix} n_{11} & \cdots & n_{1n} \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} m_{11} + n_{11} & \cdots & m_{1n} + n_{1n} \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Matrix multiplication is more complicated. The matrix product *MN* (where *M* is of dimension *m* x *n* and *N* is of dimension *p* x *q*) can be formed only if *n* = *p*, and the resulting matris is of dimension *m* x *q*. For example, if *M* is 2 x 3 and *N* is 3 x 2, then

$$MN = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{pmatrix}$$

$$= \begin{pmatrix} m_{11}n_{11} + m_{12}n_{21} + m_{13}n_{31} & m_{11}n_{12} + m_{12}n_{22} + m_{13}n_{32} \\ m_{21}n_{11} + m_{22}n_{21} + m_{23}n_{31} & m_{21}n_{12} + m_{22}n_{22} + m_{23}n_{32} \end{pmatrix},$$

of dimension 2 x 2. Thus multiplication involves calculating the dot product of each row of *M* with each column of *N*.

The multiplication of a matrix and a vector is, for example,

$$Mv = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} m_{11}v_1 + m_{12}v_2 + m_{13}v_3 \\ m_{21}v_1 + m_{22}v_2 + m_{23}v_3 \end{pmatrix}.$$

The **transpose** of a matrix is formed by swapping the rows and columns:

$$M^T = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix}^T = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \\ m_{13} & m_{23} \end{pmatrix}.$$

The **identity matrix** I_n is a square matrix of dimension *n* x *n*, with 1's on the diagonal and 0's elsewhere. For any *n* x *n* matrix *M*, $M I_n = I_n M = M$.

2.1.9 Planes and Normals

Three points (which do not all lie on the same line, i.e. are not collinear) determine uniquely a plane. We can derive the equation for the plane as follows.

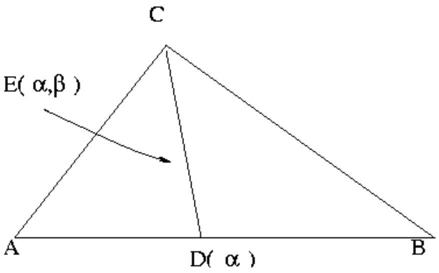

Figure 2.6

Suppose *A, B* and *C* are 3 such non-collinear points (see Figure 2.6). Then the line segment joining *A* to *B* is represented by the equation

$$D(\alpha) = \alpha A + (1-\alpha) B, \quad 0 \le \alpha \le 1.$$

For some α, draw a line from this line segment to the point *C*; this new line segment can be represented by the equation

$$\begin{aligned} E(\alpha,\beta) &= \beta D(\alpha) + (1-\beta) C, \quad 0 \le \beta \le 1 \\ &= \beta\big(\alpha A + (1-\alpha) B\big) + (1-\beta) C \\ &= \alpha\beta A + \beta B - \alpha\beta B + C - \beta C \\ &= A + \beta(1-\alpha) B + (1-\beta) C + \alpha\beta A - A + \beta A - \beta A \\ &= A + \beta(1-\alpha)(B-A) + (1-\beta)(C-A). \end{aligned}$$

So the equation for a plane can also be written in terms of a single point (e.g. *A*) and two non-parallel vectors (e.g. *B-A, C-A*). That is,

$$E(\alpha,\beta) = P_0 + \alpha u + \beta v.$$

So, for some arbitrary point P in the plane,

$$P - P_0 = \alpha u + \beta v.$$

By forming the cross product $w = u \times v$, then we have found the vector w that is perpendicular to both u and v.

So the equation of the plane can also be written

$$w \cdot (P - P_0) = 0,$$

and so w is perpendicular or normal to the plane.

Determining the normal to a plane

Given a plane defined by the equation

$$a\,x + b\,y + c\,z = d,$$

then the equation for the plane can also be written

$$n \cdot (P - P_0) = 0$$

for n the normal to the plane and P_0 a point on the plane. So we can read off the components of the normal vector $n = (a, b, c)^T$; for $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$, then

$$n \cdot (P - P_0) = a(x - x_0) + b(y - y_0) + c(z - z_0) = d - d = 0$$

so that n of this form does indeed satisfy the equation of the plane.

The normal to the plane can also be calculated from 3 points on the plane (non-collinear points). If these points are P_0, P_1, P_2 then we can calculate

$$n = (P_2 - P_0) \times (P_1 - P_0).$$

2.1.10 Matrix Properties

The following matrix properties hold, for real numbers a, b and matrices M, N, P of appropriate dimensions:

- $M + N = N + M$
- $(M + N) + P = M + (N + P)$
- $a(bM) = (ab)M$
- $a(M + N) = aM + aN$
- $(a + b)M = aM + bM$

Also, for matrices of the correct dimensions so that the following products are possible, the following hold:

- $(aM)N = a(MN)$
- $(MN)P = M(NP)$
- $(MN)^T = N^T M^T$
- $(M + N)P = MP + NP$

Note though that, in general, $MN \neq NM$ - you cannot change the order of the matrices in the product.

2.1.11 Determinant of a Matrix

The determinant of a square matrix is written *det(M)* (or $|M|$), and is defined by (for a 2 x 2 matrix)

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$

while for a 3 x 3 matrix

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a(ei - fh) - b(di - gf) + c(dh - eg).$$

2.1.12 Matrix Inverse

A square matrix M (n x n) has an inverse (written M^{-1}) if $\det(M) \neq 0$. Then $M\,M^{-1} = M^{-1}\,M = I_n$. When M represents a transformation, then M^{-1} undoes that transformation.

2.1.13 Homogeneous Coordinates

Homogeneous coordinates are used in Computer Graphics to allow consistent representation of both points and vectors and to allow the application of linear transformations (rotation, scaling, translation) using matrix multiplication.

A point in space just has a location, while a vector has direction but no location. Homogeneous coordinate systems in Computer Graphics are 4-dimensional; a 3D point is represented in homogeneous coordinates as a 4 x 1 vector, with the 4th component being a 1 in general. A 3D vector has its 4th component as a 0 when represented in homogeneous coordinates.

For example, the point at location (2, 3, 4) would be represented by $(2, 3, 4, 1)^T$, while the vector (2, 3, 4) would be written $(2, 3, 4, 0)^T$.

2.1.14 Matrix Transforms

In Computer Graphics, we often need to apply transformations to objects (defined by a set of points). These transformations can include rotating, scaling or translating, and effectively involve redrawing the object with respect to a transformed coordinate system.

Each transformation has an associated matrix, and the application of more than one transformation involves the multiplication of their respective matrices and the subsequent application of that matrix product to the object's coordinates. The order of the matrices in the matrix product is important, and similarly, so is the order of the transformations to be applied; changing the order will (usually) result in a different effect.

As well as applying transformations to an object, we have also seen transformations used to map the object (as defined in world coordinates) onto the screen/viewport. Later in this section we will describe the different sorts of mappings that can be used (for example, orthographic projections and perspective projections).

The camera position (for viewing the scene) may also need re-positioning, and so a transformation is applied in this instance too.