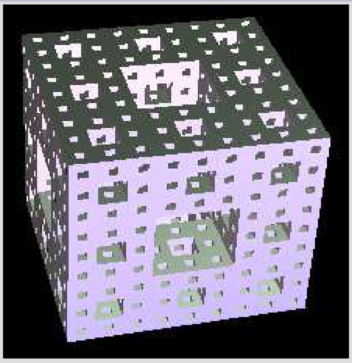
Fractals



References

http://www.math.sunysb.edu/~scott/Book331/Fractal_Dimension.html
http://www.mrob.com/pub/muency/fractaldefinitionof.html
http://hypertextbook.com/chaos/33.shtml

Overview

- Define fractal
- Recursive generation
- Language based generation

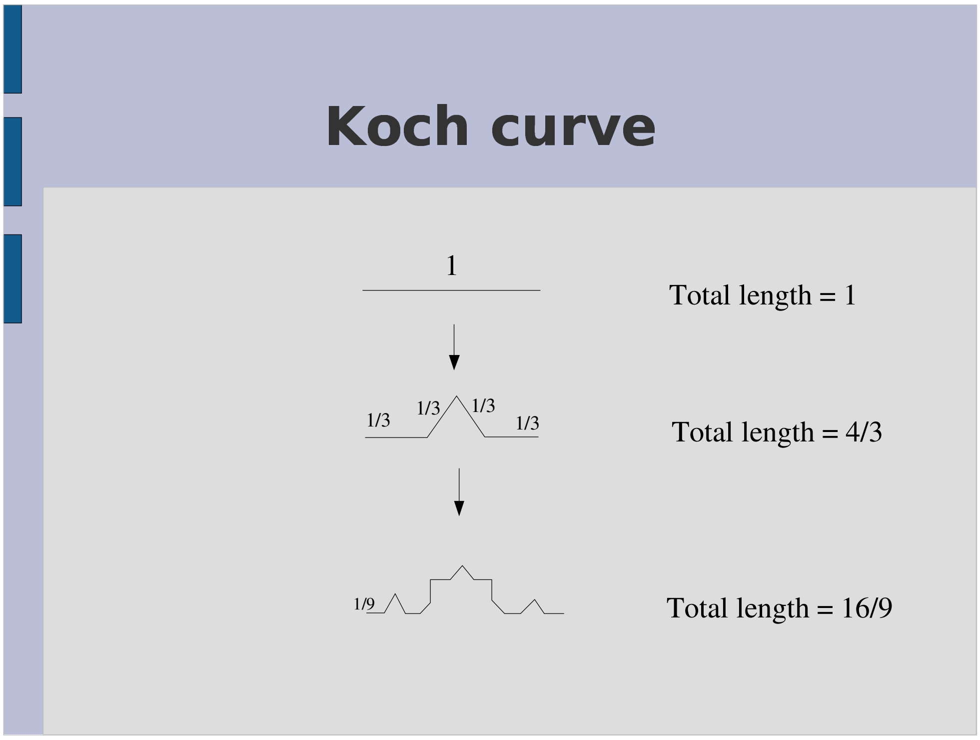
Why should I care?

- Generate realistic looking objects trading off CPU for memory
- Faster rendering through Level of Detail

/usr/X11R6/lib/xscreensaver/sierpinski3d

/usr/X11R6/lib/xscreensaver/menger
/usr/X11R6/lib/xscreensaver/lightning
/usr/X11R6/lib/xscreensaver/forest
/usr/X11R6/lib/xscreensaver/glforestfire

Show XaoS



- Interesting facts
  - Curve has infinite length but bounded by box
  - Curve never crosses itself
  - Curve is continuous
  - Curve is non differentiable everywhere
- Explain that recursive functions are a great way of doing fractals.
- Show simple recursion
- Show the Koch program

What is a Fractal?

- Coastline example
- Natural definition
- Divergent measure:
- Self-similarity
- Hausdorff definition

Benoit Mandelbrot, the discoverer of the Mandelbrot set, coined the term "fractal" in 1975 from the Latin fractus or "to break".

The oldest standard example is a coastline which when measured one kilometer at a time might turn out to be 5000 kilometers long, but when measured one meter at a time comes out to be, say, 12000 kilometers.

natural definition:

A geometric figure or natural object that combines the following characteristics: a) its parts have the same form or structure as the whole, except that they are at a different scale and may be slightly deformed; b) its form is extremely irregular or fragmented, and remains so, whatever the scale of examination; c) it contains "distinct elements" whose scales are very varied and cover a large range.

Divergent measure:

Any shape that has the unusual property that when you measure its length, area, surface area or volume in discrete finite units (as in the box-counting method), the measured value increases without finite limit as the size of the discrete unit decreases to zero.

Self-similarity

Any object that is self-similar in a non-trivial manner. An example of trivial self-similarity is a straight line: any line segment looks the same as the whole line when magnified. Non-trivial examples include such things as the Sierpinski gasket and the Koch snowflake curve.

Hausdorff definition

Any geometric form with a non-integral Hausdorff dimension.

A geometric form where the Hausdorff dimension exceeds the topological dimension.

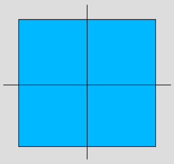
Topological Dimension

- Line = 1D
- Square = 2D
- Cube = 3D
- A manifold (set of points + coordinate systems) is n-dimensional if we need n variables to describe the local neighbourhood of the point.

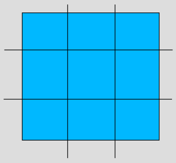
- Surface of sphere = ?D

Box Counting Dimension

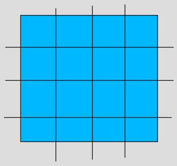
- Cover a unit square with little squares



Little square length = ½
Number of little squares = 2^2



Little square length = 1/3
Number of little squares = 3^2



Little square length = 1/4
Number of little squares = 4^2

Thus in general if we cut it into b bits along and axis then

Little square length = 1/b
N = Number of little squares = b^2

Box Counting Dimension continued

- Consider the unit line segment
  - Little line length = 1/b
  - N = Number of little lines = b^1
- Consider the unit cube
  - Little cube length = 1/b
  - N = Number of little cubes = b^3
- So for the d^{th} dimension : $N = b^d$

Fractal Dimension

- $N = b^d$
- Thus

$$d = \frac{\ln N}{\ln b}$$

Where

- N is the total number of similar objects we create by subdivision and
- b is the number of bits the axis is subdivided into. Note that b is a scaling factor.

Koch dimension = ln 4 / ln 3 = 1.26186

So does the Koch dimension of 1.26... make sense.
Dimension is obviously 1 until we iterate an infinite number of times.
Length is then infinite and derivative is discontinuous everywhere, thus isn't 1D. Not 2D as it is infinitely “holey”.
Thus somewhere between 1 and 2 makes sense.

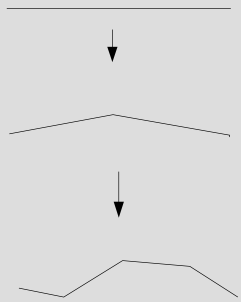
Why should I care?

- Fractal curve has 1<= d < 2
- Fractal surface has 2<= d < 3

- Dimension is related to roughness!
- Lower dimensions are smoother than higher dimensions
- E.g. Mountain will have higher dimension that farmland.

Midpoint division

- Similar idea to Koch curve



- Mountain generated on 2D mesh with midpoint division.

Language Method

- A → BC
- B → ABA

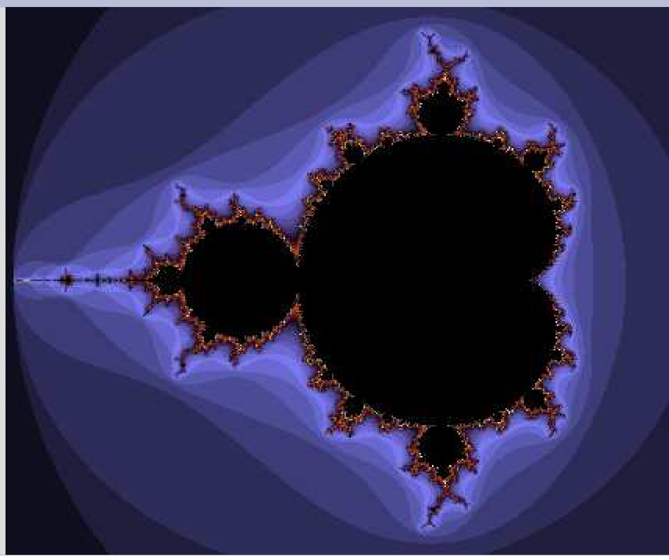
- CAB
  - CBCABA
  - CABACBCABABC

- Koch rule
    - Turtle graphics
      - F = forward
      - L = turn left 60 degrees
      - R = turn right 60 degrees
    - F -> FLFRRFLF

Push and Pop Operators

- If the language is expanded to allow push and pop operators – represented by [and]
- F -> F[RF]F[LF]F
  - F
  - F[RF]F[LF]F
  - F[RF[RF]F[LF]F]F[RF]F[LF]F[LF[RF]F[LF]F]F[RF]F[LF]F
- Generates same fractal every time
- Introduce randomness

Mandelbrot Set



Show xaos

Complex Number

- $\mathbf{z} = x + \mathbf{i}\, y$
- $\mathbf{i}^2 = -1$
- $|z| = x^2 + y^2$
- $\mathbf{w} = F(\mathbf{z})$
- $\mathbf{z}_{k+1} = F(\mathbf{z}_k)$
- $\mathbf{z}_{k+1} = \mathbf{z}_k^2$ where $\mathbf{z}_0 = c$
- $\mathbf{z}_{k+1} = \mathbf{z}_k^2 + c$ where $\mathbf{z}_0 = 0 + \mathbf{i}\, 0$

First recursion goes to attractor at origin for abs(z) < 1 goes infinite for abs(z)>1 and abs(z) = 1 bounces around the unit circle.

Second recursion is the Mandelbrot set. The point c is in the Mandelbrot set iff the the series remains finite. At edges of Mandelbrot set you can't trivially compute whether the series is finite or not. Compute to some iteration and then assign colour based on final value of z.