COMS3100/7100

Introduction to Communications

Lecture 7: *Correlation & Spectral Density*

This lecture:

1. Correlation of Energy Signals
2. Energy Spectral Density
3. Correlation of Power Signals
4. Power Spectral Density
5. Properties of Correlation & Spectral Density

Ref: CCR pp. 124–135, Couch pp. 61–65.

Correlation of Energy Signals

Recall that a signal $x(t)$ is called an *energy signal* if $E\{x(t)\} < \infty$.

- We define the *(cross-)correlation* of two real-valued energy signals $x(t)$ and $y(t)$ as

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t-\tau)\,dt.$$

- The parameter τ is termed the *time difference* or *lag*.
- The correlation measures how similar $x(t)$ and $y(t)$ are at lag τ.
  - A large value indicates a high degree of similarity.
- For complex-valued energy signals, we define correlation as

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y^*(t-\tau)\,dt.$$

- Observe the similarity with convolution. In fact,

$$R_{xy}(\tau) = x(\tau) * y^*(-\tau).$$

Energy Cross-Spectral Density

Consider the Fourier transform of $R_{xy}(\tau)$ (taken over τ instead of the usual t).

- We call this the *energy cross-spectral density* $G_{xy}(f)$.
- From the convolution & time-reversal properties of the Fourier transform, we have

$$G_{xy}(f) = X(f)Y^*(f).$$

Autocorrelation

We can calculate the correlation of a signal with itself.

- This is *autocorrelation* which, for an energy signal, is

$$R_x(\tau) = x(\tau) * x^*(-\tau).$$

⏮ ◀ ▶ ⏭ ⏪ ⏩ ⏸

Energy Spectral Density

It follows that we can similarly define the *energy spectral density (ESD)* $G_x(f)$ as the Fourier transform of $R_x(\tau)$ so that

$$E_x(f) = X(f)X^*(f) = |X(f)|^2.$$

- The name is explained by application of Parseval's theorem:

$$E\{x(t)\} = \int_{-\infty}^{\infty} |X(f)|^2 \, df = \int_{-\infty}^{\infty} G_x(f) \, df.$$

- Observe also that

$$E\{x(t)\} = \int_{\infty}^{\infty} |x(t)|^2 \, dt = R_x(0).$$

Correlation of Power Signals

Recall that a signal $x(t)$ is called a *power signal* if $P\{x(t)\} < \infty$.

- For power signals, we require a slightly different definition of correlation since the definition for energy signals doesn't converge.
- Instead, we define the cross-correlation of two power signals $x(t)$ and $y(t)$ in terms of (asymptotic) time averages:

$$R_{xy}(\tau) = \langle x(t)y^*(t-\tau)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)y^*(t-\tau)\,dt.$$

- Similarly, autocorrelation of a power signal $x(t)$ is defined as

$$R_x(\tau) = \langle x(t)x^*(t-\tau)\rangle .$$

Power Spectral Density

A signal with non-zero power is (by definition!) not square integrable, so we can't compute its Fourier transform.

- Hence, the spectral density defined for energy signals is not applicable to power signals either, so we take another approach.
- Defining $x_T(t) = \Pi(t/T)x(t)$, the power can be computed as

$$P\{x(t)\} = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x_T(t)|^2 \, dt.$$

- By Parseval's equality, it follows that

$$P\{x(t)\} = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |X_T(f)|^2 \, df.$$

- Exchanging limits & integration above, we define the *power spectral density* (PSD) of $x(t)$ as

$$G_x(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}. \tag{1}$$

The Wiener-Khintchine Theorem

The *Wiener-Khintchine theorem* states that

$$R_x(\tau) \xleftrightarrow{FT} G_x(f).$$

- This is the same relationship that holds for energy signals

We can calculate the PSD in two different ways:

1. *Direct method*, using (1) on the previous slide.
2. *Indirect method*, by calculating the autocorrelation first, then FT.

Power Cross-Spectral Density

Similarly, it's possible to define a *power cross-spectral density* of two power signals $x(t)$ and $y(t)$ as

$$G_{xy}(f) = \lim_{T\to\infty} \frac{X_T(f)Y_T^*(f)}{T}.$$

- The Wiener-Khintchine theorem can be generalised, so that

$$R_{xy}(\tau) \xleftrightarrow{FT} G_{xy}(f).$$

Properties of Correlation & Spectral Density

The following additional properties apply to correlation and spectral density:

1. $R_x(0) = E\{x(t)\}$ for energy signals and $R_x(0) = P\{x(t)\}$ for power signals.
2. $\left|R_{xy}(\tau)\right|^2 \leqslant E\{x(t)\}E\{y(t)\}$ for energy signals and $\left|R_{xy}(\tau)\right|^2 \leqslant P\{x(t)\}P\{y(t)\}$ for power signals.
3. $|R_x(\tau)| \leqslant R_x(0)$.
4. $R_{xy}(-\tau) = R^*_{yx}(\tau)$.
5. $R_x(-\tau) = R^*_x(\tau)$.
6. $G_x(f)$ is real.
7. $G_x(f) \geqslant 0$.
8. When $x(t)$ is real, $G_x(f)$ is even.

LTI Systems

There are some additional properties that apply when $x(t)$ is the input to an LTI system and $y(t)$ is the corresponding output.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

1. $R_{xy}(\tau) = h(\tau) * R_x(\tau)$.
2. $R_y(\tau) = h^*(-\tau) * h(\tau) * R_x(\tau)$.
3. $G_{xy}(f) = H(f)G_x(f)$.
4. $G_y(f) = |H(f)|^2 G_x(f)$.