Advanced Computational Techniques in Engineering

Lecture SP01: *Probability*

This lecture:

1. Probability.
2. Repeated Trials.

Ref: STAT2202 course notes, PP ch. 2–3.

ENGG7302

Probability

Recall that a *probability space* is the triple $(\Omega, \mathscr{A}, P)$.

- Ω is a *sample space* and corresponds to all possible *outcomes*.
- $\mathscr{A}$ is the *σ-algebra* of Ω, which means that $\mathscr{A}$ is a non-empty collection of subsets of or *events* in Ω such that:
  - $\Omega \in \mathscr{A}$,
  - if a set $S \in \mathscr{A}$ then so is its complement,
  - if S_n is a sequence in $\mathscr{A}$, so is their union.
- P is a *probability measure* on $\mathscr{A}$ which means:
  - $P : \mathscr{A} \mapsto \mathbb{R}$,
  - $0 \leqslant P(S) \leqslant 1$,
  - $P(\emptyset) = 0$,
  - $P(\Omega) = 1$,
  - for any countable collection of pairwise disjoint set S_n,

$$P(\bigcup_n S_n) = \sum_n P(S_n).$$

Conditional Probability

Suppose that we know that some event $\mathcal{B}$ has taken place.

- What is the probability now that event $\mathcal{A}$ also occurs?
- This is the *conditional probability of $\mathcal{A}$ given $\mathcal{B}$* and

$$P(\mathcal{A} \mid \mathcal{B}) = \frac{P(\mathcal{A} \cap \mathcal{B})}{P(\mathcal{B})}. \tag{1}$$

- We often abbreviate $\mathcal{A} \cap \mathcal{B}$ as $\mathcal{A}\mathcal{B}$.

Chain Rule

Equation 1 can be rewritten as

$$P(\mathcal{A}\mathcal{B}) = P(\mathcal{A} \mid \mathcal{B})P(\mathcal{B}). \tag{2}$$

- This can be generalised to give the *chain rule*:

$$P(\mathcal{A}_1\mathcal{A}_2 \cdots \mathcal{A}_n) = P(\mathcal{A}_1 \mid \mathcal{A}_2 \cdots \mathcal{A}_n) \cdots P(\mathcal{A}_{n-1} \mid \mathcal{A}_n)P(\mathcal{A}_n)$$

for events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$.

Bayes' Rule

The events $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are a *partition* of Ω if they are mutually disjoint and

$$\Omega = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n.$$

- By appeal to (2), we find that

$$P(\mathcal{A}) = \sum_{i=1}^{n} P(\mathcal{A} \mid \mathcal{B}_i) P(\mathcal{B}_i),$$

  which is the *total probability theorem*.

- Observing from (2) that

$$P(\mathcal{A}\mathcal{B}_j) = P(\mathcal{A} \mid \mathcal{B}_j) P(\mathcal{B}_j) = P(\mathcal{B}_j \mid \mathcal{A}) P(\mathcal{A}),$$

  application of the total probability theorem yields *Bayes' rule*:

$$P(\mathcal{B}_j \mid \mathcal{A}) = \frac{P(\mathcal{A} \mid \mathcal{B}_j) P\left(\mathcal{B}_j\right)}{\sum_{i=1}^{n} P(\mathcal{A} \mid \mathcal{B}_i) P(\mathcal{B}_i)}.$$

Independence

The events $\mathcal{A}$ and $\mathcal{B}$ are *independent* if and only if

$$P(\mathcal{A}\mathcal{B}) = P(\mathcal{A})P(\mathcal{B}).$$

- *Mutual independence* of events $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is defined recursively:
  - Any (strict) subset of the events are mutually independent and
  - $P(\mathcal{A}_1 \cdots \mathcal{A}_n) = P(\mathcal{A}_1) \cdots P(\mathcal{A}_n)$.

Repeated Trials

Cartesian Product

Suppose two experiments are to be performed.

- The outcomes of the first experiment belong to a set Ω_1.
- Those of the second belong to Ω_2.
- The set of all possible outcomes of the two experiments is a set of ordered pairs Ω, the *Cartesian product of Ω_1 and Ω_2*:

$$\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) \mid \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}.$$

Bernoulli Trials

Suppose a number of experiments are to be performed but we are only interested in whether or not a particular event occurs in each.

- This we call *Bernoulli trials*.

- An important special case is where the experiments are identical and independent.
- Let $\mathcal{A}$ be the event of interest in each trial and set $p = P(\mathcal{A})$.
- Let $\mathcal{B}$ be the event that $\mathcal{A}$ occurs exactly k times in n trials in any order.
- The *binomial law* states that

$$P(\mathcal{B}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

- Let $\mathcal{G}$ be the event that $\mathcal{A}$ *first* occurs in the kth trial.
- The *geometric law* states that

$$P(\mathcal{G}) = (1-p)^{k-1} p.$$