Possible World Semantics for Quantified Non-normal Modal Logics

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ML + FOL $\neq$ QML

- $(\Box, \Diamond) + (\wedge, \vee, \neg, \rightarrow) =$ Modal Logic
- Possible World Semantics $\approx$ set of propositional interpretations
- $(\Box, \Diamond) + (\wedge, \vee, \neg, \rightarrow) + (\forall, \exists) \neq$ Quantified Modal Logic
- Possible World Semantics $\approx$ set of first order interpretations

Normal Modal Logics

- Your preferred set of axioms for standard propositional logic
- $\Box \equiv \neg\Diamond\neg$
- $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- $\dfrac{\vdash A}{\vdash \Box A}$

Non-normal Modal Logics

- Minimal non-normal modal logic (E)
  - Your preferred set of axioms for standard propositional logic
  - $\Box \equiv \neg\Diamond\neg$
  - $\dfrac{\vdash A \equiv B}{\vdash \Box A \equiv \Box B}$
- M: $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$
- C: $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$
- N: $\Box\top$

Quantified (Non-) Normal Modal Logic

- Your preferred set of axioms for standard propositional logic

Quantified (Non-) Normal Modal Logic

- Your preferred set of axioms for standard propositional logic
- $\forall x A(x) \rightarrow A(x)$
- $\forall x (A \rightarrow B(x)) \rightarrow (A \rightarrow \forall x B(x))$
- $\dfrac{A(x)}{\forall x A(x)}$

Quantified (Non-) Normal Modal Logic

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- $\forall x(A \rightarrow B(x)) \rightarrow (A \rightarrow \forall x B(x))$
- $\dfrac{A(x)}{\forall x A(x)}$
- Your preferred modal logic

Quantified (Non-) Normal Modal Logic

- Your preferred set of axioms for standard propositional logic
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- Your preferred modal logic (NO!!!)

Quantified (Non-) Normal Modal Logic

- Your preferred set of axioms for standard propositional logic
- $\forall x A(x) \rightarrow A(x)$
- $\forall x(A \rightarrow B(x)) \rightarrow (A \rightarrow \forall x B(x))$
- $\dfrac{A(x)}{\forall x A(x)}$
- Your preferred modal logic (NO!!!)
- *BF*: $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- *CBF*: $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

Possible Worlds Semantics

$$\mathcal{M} = \langle W, X, D, d, I \rangle$$

- W is a set of possible worlds
- X is ?
- D set of individuals
- d is function from W to 2^D (it gives the domain of w)
- I is an interpretation function such that:
  - $I(x, w) \in D$ (global interpretation of variables/terms);
  - $\forall w, w' \in W, I(x, w) = I(x, w')$ (rigidity of variables/terms);
  - $I(\phi(x_1, \ldots, x_n), w) \subseteq D^n$.
- $\|A\|_I^{\mathcal{M}} = \{w : \mathcal{M}, w \models_I A\}$

$\mathcal{M}, w \models_I \forall x A(x)$ iff for every $d \in d(w)$, $\mathcal{M}, w \models_{I_{d/x}} A(x)$, where $I_{d/x}$ is like I except for mapping x to d.

Kripke Models I (Normal Modal Logics)

$$\mathcal{K} = \langle W, R, D, d, I \rangle$$

- $R \subset W \times W$ the accessibility relation
- $\mathcal{K}, w \models_I \Box A$ iff $\forall w'(wRw'), \mathcal{K}, w' \models_I A$

Conditions on models

BF	If wRw' then $d(w') \subseteq d(w)$ (decreasing domains)
CBF	If wRw' then $d(w) \subseteq d(w')$ (increasing domains)
FOL	$\forall w, w'\, d(w) = d(w') = D$. (constant domains)

Kripke Models II

Theorem

- *BF and CBF are valid.*
- *The smallest quantified normal modal logic is*
  - *FOL*
  - $\Box \equiv \neg\Diamond\neg$
  - $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
  - $\dfrac{\vdash A}{\vdash \Box A}$
  - *BF:* $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$

Neighbourhood Models

Minimal models, Scott-Montague models

$$\mathcal{N} = \langle W, N, D, d, I \rangle$$

- $N : W \mapsto 2^{2^W}$, N maps every world to a set of sets of worlds (propositions)
- $\mathcal{N}, w \models_I \Box A$ iff $\|A\|_I^{\mathcal{N}} \in N_w$

Conditions on models

M $\forall w \in W, X \cap Y \in \mathcal{N}_w \Rightarrow X \in \mathcal{N}_w$ and $Y \in \mathcal{N}_w$ (supplemented)

C $\forall w \in W, X \in \mathcal{N}_w$ and $Y \in \mathcal{N}_w \Rightarrow X \cap Y \in \mathcal{N}_w$ (closed under intersection)

BF $\forall w \in W, \emptyset \subset \mathcal{X} \subseteq \mathcal{N}_w \Rightarrow \bigcap_{X \in \mathcal{X}} X \in \mathcal{N}_w$ (closed under arbitrary intersection)

N For every w, $W \in N_w$. (each N_w contains the unit)

FOL $\forall w, w'\ d(w) = d(w') = D$ (constant domains)

Neighbourhood Models II

Theorem (Arló-Costa 2005)

The smallest quantified non-normal modal logic is

- *FOL*
- $\Box \equiv \neg\Diamond\neg$
- $\dfrac{A \equiv B}{\Box A \equiv \Box B}$

Neighbourhood Models III

Theorem (Waagbø 1992)

- *M is valid in the class of supplemented models*
- *CBF is valid in the class of supplemented models*
- *C is valid in the class of models closed under intersection*
- *BF is valid in the class of models closed under arbitrary intersection*

Finite Neighbourhood Models

Theorem (Arló-Costa 2002)

If a finite constant domain neighbourhood frame is closed under intersection then BF and CBF are valid in every model based on the frame.

Regular Neighbourhood Models

A neighbourhood model is regular iff

1. $\forall w, N_w \neq \emptyset$
2. $\forall w, \emptyset \notin N_w$
3. $|D| > 1$

1. there is at least one formula $\Box A$ true for every world
2. $\neg\Box\bot$, internal consistency of $\Box$
3. there are at least two individuals/things

Filters

Given a set of sets Γ,

- Γ is a quasi-filter if supplemented (or closed under supersets) and closed under intersection
- Γ is a filter if is a quasi-filter and it contains the unit (or if it is a non-empty quasi-filter)
- Γ is augmented if $X \in \Gamma$ iff $\bigcap \Gamma \subseteq X$.
- finite filters are augmented

Filters and Regular Models

Theorem (Arló-Costa 2002, Stolpe 2003)

- *A regular and finite constant domain neighbourhood frame is a quasi-filter if and only if BF and CBF are valid in every model based on it.*
- *A regular and finite constant domain neighbourhood frame is a filter if and only if BF and CBF are valid in every model based on it.*
- *A regular and finite constant domain neighbourhood frame is augmented if and only if BF and CBF are valid in every model based on it.*

Theorem

- *K is characterised by the class of filters.*
- *K is characterised by the class of augmented frames.*

Multi-Relational Models (Governatori & Rotolo 2005)

$$\mathcal{R} = \langle W, R^*, D, d, I \rangle$$

- $R^* \subseteq 2^{W \times W}$, R^* is a set of of binary relations over W
- $\mathcal{R}, w \models_I \Box A$ iff $\exists R \in R^*$ such that $\forall w'(wRw'$ iff $\mathcal{R}, w' \models_I A)$

Conditions on models

M let $T \subseteq W$, $\forall w \in N \forall R \in R^*$ if $R_w \subseteq T$, then $\exists S \in R^*$ such that $S_w = T$ (R^* is point-wise closed under supersets)

C $\forall w \in N \forall R, S \in R^* \exists T \in R^*$ such that $R_w \cap S_w = T_w$ (R^* is point-wise closed under intersection)

Multi-Relational Models II

Theorem

- *M, C, BF are not valid in the class of all multi-relational models*
- *CBF is valid in the class of all multi-relational models.*

Multi-Relational Models III

Theorem

M *is characterised by the class of models point-wise closed under supersets*

C *is characterised by the class of models point-wise closed under intersection*

BF *is characterised by the class of multi-relational models where R^* is a singleton (monadic models)*

Corollary

Every monadic model is point-wise closed under intersection

Comparisons

Neighbourhood Models

- Smallest logic has no modal/quantification principles (BF, CBF)
- $CBF \Leftrightarrow M$
- $BF \Rightarrow C$ and $BF \Leftrightarrow C$ for finite models
- for finite models C implies collapse into normal modal logic.

Multi-Relational Models

- Smallest logic requires CBF
- M is independent from CBF
- $BF \Rightarrow C$
- $C \not\Rightarrow BF$
- No collapse to normal modal logic

Conclusions (Neighbourhood Models)

- Non-normal modal logics have been proposed as an alternative for interpretations of $\Box$ where normal modal logics are a too rough approximation (deontic, epistemic, agency, ability, ...)
- Usually the semantics is given in terms of neighbourhood models
- For propositional modal logic neighbourhood models have nice mathematical properties
- However, it lacks a good intuitive explanatory of the structure.
- Serious shortcoming for quantified modal logic based on standard first-order logic: Quantificational principles have strong effects on modal principles and the other way around.
- Lack of intuitive reading for *BF* and *CBF*

Conclusions (Multi-Relational Models)

- Claimed isomorphism with neighbourhood models (at least for propositional non-normal modal logic), but obviously non-isomorphic for quantified modal logic!
- Intuitive readings of models and *BF*, *CBF*
- Limited dependence of modal principles from quantificational principles for standard first order logic.

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