

On the complexity of the extended Euclidean algorithm

(Extended Abstract)

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Euclid's algorithm for computing the greatest common divisor of 2 numbers is considered to be the oldest proper algorithm known ([10]). This algorithm can be amplified naturally in various ways. The GCD problem for more than two numbers is interesting in its own right. Thus, we can use Euclid's algorithm recursively to compute the GCD of more than two numbers. Also, we can do a constructive computation, the so-called extended GCD, which expresses the GCD as a linear combination of the input numbers.

Extended GCD computation is of particular interest in number theory (see [1, chapters 2 and 3]) and in computational linear algebra ([3, 4, 9]), in both of which it takes a basic role in fundamental algorithms. An overview of some of the earlier history of the extended GCD is given in [1], showing that it dates back to at least Euler.

Motivated by many efforts to find good algorithms for extended GCD computation, Majewski and Havas ([12]) showed that this is genuinely difficult. There are a number of problems for which efficient solutions are not readily available.

Theorem 1.

Given set A of positive integers and positive integer K , the problem "Does A contain a subset R such that $|R| \leq K$ and $\gcd(r \in R) = \gcd(a \in A)$?" is NP-complete.

Theorem 2.

Given set A of positive integers and positive integer K , the problem "Does

there exist a set of multipliers $\{x_i\}$ such that no x_i is larger in magnitude than K and $\sum_{a_i \in A} x_i a_i = \gcd(a \in A)$?" is NP-complete.

On the other hand, if we are willing to accept an upper bound on the quality of the solution which depends on the size of the input numbers instead of on the size of the optimal solution we observe the following result.

Theorem 3.

There is an optimal time, optimal space algorithm for computing the extended GCD of n integers $\{a_1, \dots, a_n\}$, which guarantees that no multiplier is larger than the largest of the numbers divided by 2.

Even though Theorem 1 tells us that finding a sparsest possible solution to the extended GCD problem is NP-complete, the problem is almost always easy ([5]). There exists a class of problems, AP, which, although NP-complete, have average polynomial time complexity. This problem is in AP.

Theorem 3 can be improved as follows [2].

Theorem 4.

Let $\{a_1, \dots, a_n\}$ be a multiset of positive integers. Let $m = \max(a_1, \dots, a_{n-1})$ and let $g_k = \gcd(a_k, \dots, a_n)$ for $1 \leq k \leq n$. Then there exists an integer solution to the equation

$$x_1 a_1 + \dots + x_n a_n = g_1$$

which satisfies

$$-\frac{g_{j+1}}{2g_j} < x_j \leq \frac{g_{j+1}}{2g_j} \text{ for } 1 \leq j \leq n-1, \tag{1}$$

$$|x_n| \leq \max\left(\frac{m}{2g_1}, 1\right). \tag{2}$$

Theorem 4 is realized by a practical algorithm. Furthermore, it is optimal in the sense that the bounds given in the Theorem for the multipliers can be approached as closely as desired by any individual multiplier. It is not worse than about the square of the best possible bound for distinct numbers, since a general lower bound for the Euclidean norm of the multiplier vector in terms of the initial numbers a_i must be at least $O(\sqrt{\max\{a_i\}})$ (see [6]).

Let us return to the hard problems. One way of addressing hard problems is to seek approximately optimal solutions instead of optimal solutions (see [8]). Consider the ℓ_p -norm shortest extended GCD multiplier problem [7]:

SHORTEST GCD MULTIPLIER in ℓ_p -norm (SGCDM _{p})

Instance: n numbers $a_1, \dots, a_n \in \mathbb{Z}$

Objective: Find a vector $\mathbf{x} \in \mathbb{Z}^n$ such that $\sum_{i=1}^n x_i a_i = \gcd(a_1, \dots, a_n)$ and such that the ℓ_p -norm $\|\mathbf{x}\|_p := (\sum_{1 \leq i \leq n} |x_i|^p)^{1/p}$ of the vector \mathbf{x} is minimized.

We have the following results:

Theorem 5.

1. *Unless $\text{NP} \subseteq \text{P}$, there exists no polynomial-time algorithm which approximates the SHORTEST GCD MULTIPLIER problem in ℓ_p -norm within a factor of k , where $k \geq 1$ is an arbitrary constant.*
2. *Unless $\text{NP} \subseteq \text{DTIME}(n^{\text{poly}(\log n)})$, there exists no polynomial-time algorithm which approximates the SHORTEST GCD MULTIPLIER problem in ℓ_p -norm within a factor of $n^{1/(p \log^\gamma n)}$, for γ an arbitrary small positive constant.*

Corollary 6.

For every ℓ_p -norm, the Extended GCD problem is NP-complete (in its feasibility recognition form).

How well can we approximate optimal solutions in polynomial time? Approximating the solution to the ℓ_2 -norm problem within a factor of $\sqrt{n/O(\log n)}$ is very unlikely NP-hard.

Theorem 7.

The $\sqrt{n/O(\log n)}$ -SHORTEST GCD MULTIPLIER in ℓ_2 -norm is not NP-hard unless the Polynomial-Time Hierarchy collapses to its second level.

Using lattice basis reduction [11] an algorithm HMM has been developed and analyzed [6, 13]. It is polynomial time and achieves a factor of $2^{(n-1)/2}$.

Theorem 8.

On inputs $a_1, \dots, a_n \in \mathbb{Z}$ algorithm HMM computes in polynomial time a vector \mathbf{x} with $\langle \mathbf{x}, \mathbf{a} \rangle = \gcd(\mathbf{a})$ satisfying

$$\|\mathbf{x}\| \leq 2^{(n-1)/2} \cdot \text{opt}_{\text{SGCDM}_2}(a_1, \dots, a_n).$$

It remains to be seen whether this substantial gap between around \sqrt{n} and $2^{(n-1)/2}$ can be reduced.

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