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Letter to Editor

The fourth moment of the radial displacement of a discrete correlated/persistent random walk

Dear Editor,

An exact closed form solution to the fourth radial moment, $\langle R^4 \rangle$, of a correlated/persistent random walk is reported here, where the expected mth radial displacement $\langle R^m \rangle = \int_0^\infty R^m f(R)dR$, and $f(R)$ is the probability density function of the radial displacement R. This is the analytic expression for the true mean of the 4th power of the net distance from the origin following a correlated/persistent random walk. The solution reported here contradicts a previously reported result which appears to be mathematically incorrect.

Correlated/persistent random walk models have been used to describe a wide range of phenomena including insect movements (Karieva and Shigesada, 1983), population dynamics (Skellam, 1973), foraging patterns (Root and Karieva, 1984; Cain, 1985; Turchin, 1991), animal dispersal (Kitching, 1971; Byers, 2000), spread of plants (Okubo and Levin, 1989), cell migration (Nossal and Weiss, 1974), microbial transport (Li et al., 1996), even polymer chain dynamics (Tchen, 1952, Bracher, 2004). For more extensive reviews, see Berg (1983), Weiss (1994) or Codling et al. (2008).

The key advancement of correlated/persistent random walk models over the standard random walk (Pearson, 1905) or Brownian motion (Einstein, 1905), lies in large part on introducing statistical correlation between spatiotemporally adjacent steps/links along a trajectory/chain. A commonly used formulation of this first-order correlation has been described in detail previously (Bovet and Benhamou, 1988; Cheung et al., 2007). Briefly, the correlation is generated by defining a random distribution of turn angles, Δ, between successive steps. Turn angles are assumed to be statistically independent of each other and of step size (which can follow any distribution). Otherwise, any unbiased distribution of Δ is valid (which includes all symmetrical distributions). In the limit, a uniform distribution of Δ results in no correlation and hence a standard random walk. The term "persistence" refers to the tendency to remain oriented in a similar direction for temporally close steps (i.e. directional correlation), and governs the straightness of individual trajectories/chains (Cheung et al., 2007).

An exact closed form expression of the first moment of the radial displacement, $\langle R \rangle$, of a correlated/persistent random walk is still unavailable. However, it can be shown that as the number of steps become asymptotically large, $\lim_{n\to\infty} \langle R \rangle = \sqrt{\pi \langle R^2 \rangle}/2$ (Bovet and Benhamou, 1988; Benhamou, 2004). For smaller values of step number n, McCulloch and Cain (1989) provided a closer approximation of $\langle R \rangle$, expressed in terms of both $\langle R^2 \rangle$ and $\langle R^4 \rangle$ (see later).

Interestingly, the second radial moment, $\langle R^2 \rangle$, has been known for some time (Karieva and Shigesada, 1983). One of the most important reasons that the radial displacement is considered in many applied studies of animal/plant movement on the 2D plane is that *a priori*, it is impossible to know the goal direction of the dispersing agent. In a correlated/persistent random walk model, the radial displacement is the same irrespective of the initial orientation of the individual, or population of dispersing agents. This is distinct from the situation when there is a known initial direction (hence a directed walk), where exact first and second positional moments are available, calculated along and perpendicular to this axis of "intended" locomotion (Cheung et al., 2007; Cheung et al., 2008). This is also distinct from the so-called Biased Random Walks (BRWs) which result from a global directional bias (see explanation in Codling, 2008), which may arise, for instance, in an animal using a compass cue to maintain orientation whilst attempting to move in a straight line (e.g. an allothetic directed walk—see Cheung et al., 2007; Cheung et al., 2008).

The 4th radial moment $\langle R^4 \rangle$ is useful to obtain the exact variance of the squared radial dispersal, $V(R^2) = \langle R^4 \rangle - \langle R^2 \rangle^2$, of a correlated/persistent random walk. Moreover, McCulloch and Cain (1989) used the first few terms of a Taylor series expansion of the square root function, combined with $V(R^2)$ to get $\langle R \rangle \approx \sqrt{\langle R^2 \rangle}\{1 - \frac{1}{8}V(R^2)/\langle R^2 \rangle^2\}$. Hence an exact $\langle R^4 \rangle$ has direct use in quantifying the variability of R^2 but also in estimating $\langle R \rangle$ and $V(R)$.

Higher moments of the radial displacement have been reported by Claes and Van den Broeck (1987), and apparently obtained via a symbolic manipulator. Their model was aimed at describing the end-to-end distance of polymer chains, and the persistence was formulated as follows: molecular segment k (analogous to step k) had the same orientation as segment $k-1$ with probability p, and uniformly random orientation otherwise, giving a symmetrical distribution of the difference in segment directions, Δ. Although this distribution is not smooth and continuous like the Gaussian function (the centre of the density function is p times a Dirac's delta function, and the rest is a uniform distribution with density $(1-p)/2\pi$)) it nevertheless follows all the assumptions of a correlated/persistent random walk. Using the formulation of a correlated random walk, the correlation parameter is

$$\beta = \langle \cos(\Delta) \rangle = p\cos(0) + \frac{(1-p)}{2\pi}\int_{-\pi}^{+\pi} \cos(\theta)d\theta = p \tag{1.1}$$

Hence the 4th radial moment reported by Claes and Van den Broeck (1987) is

$$\langle R^4 \rangle = \frac{5}{3}n^2\left(\frac{1+\beta}{1-\beta}\right)^2 + \frac{16}{3}n^2\frac{\beta^{n+1}}{(1-\beta)^2} + 4n\beta^{n+1}\frac{1+\beta}{(1-\beta)^3} + \frac{8}{3}\frac{1-\beta^n}{(1-\beta)^4}(\beta^3+\beta^2+\beta) - \frac{2}{3}n\frac{1+\beta}{(1-\beta)^3}(\beta^2+10\beta+1) \tag{1.2}$$

if an unitary step/link length is assumed. (In the original form, the authors assumed a constant step/link length, b, so for any $\langle R^m \rangle$

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the solution is scaled by b^m.) However, this equation is generally incorrect and deviates significantly with Monte Carlo results (see Figs. 1 and 2).

A slightly lengthier, correct expression is given below, and its derivation briefly outlined (see Appendix A for details). In summary, we expand X^2+Y^2, collect $\sin^2(\cdot)+\cos^2(\cdot)$ terms, apply the identity $\cos(u)\cos(v)+\sin(u)\sin(v)=\cos(u-v)$, and substitute $\Theta_j-\Theta_i=\Delta_{i+1}+\cdots+\Delta_j$ to get

$$\langle R^4\rangle = \langle (X^2+Y^2)^2\rangle = \ldots = n^2\left(\frac{1+3\beta}{1-\beta}\right)-n\frac{4\beta(1-\beta^n)}{(1-\beta)^2}+4Q \quad (1.3)$$

where n is the step number, $\Theta_j=\sum_{w=1}^{j}\Delta_w$ is the absolute direction at step j. This substitution is made to yield terms of the form $\cos(\Delta_i+\cdots+\Delta_n)$ which are simpler to evaluate because of the assumed statistical independence of successive Δ. Also, we defined

$$Q=\sum_{j=2}^{n}Q_{jj}+2\sum_{i=2}^{n-1}\sum_{j=i+1}^{n}Q_{ij} \quad (1.4)$$

where

$$Q_{ij}=\langle(\cos(\Delta_i)+\cdots+\cos(\Delta_i+\cdots+\Delta_n))(\cos(\Delta_j)+\cdots+\cos(\Delta_j+\cdots+\Delta_n))\rangle \quad (1.5)$$

By expanding and then carefully summing all the geometric series, it can be shown that

$$\sum_{j=2}^{n}Q_{jj}=\frac{\beta(2\gamma-1)}{(1-\beta)(2\gamma-1-\beta)}\left(\frac{\beta-\beta^n}{1-\beta}-\frac{(2\gamma-1)-(2\gamma-1)^n}{2-2\gamma}\right) -\frac{(1+\beta)\left((2\gamma-1)^2-(2\gamma-1)^{n+1}\right)}{8(1-\beta)(1-\gamma)^2}+\frac{(n-1))(1+\beta)(2\gamma-1)}{4(1-\beta)(1-\gamma)} +\frac{\beta^2-\beta^{n+1}}{(1-\beta)^3}-\frac{(n-1)\beta}{(1-\beta)^2}+\frac{n(n-1)(1+\beta)}{4(1-\beta)} \quad (1.6)$$

where $\gamma=\langle\cos^2\Delta\rangle$. More tediously, but using much the same techniques, it can be shown that

$$\begin{aligned}\sum_{i=2}^{n-1}\sum_{j=i+1}^{n}Q_{ij} &= \sum_{i=2}^{n-1}\sum_{j=i+1}^{n}\frac{\beta^2(1-\beta^{n-j+1})(1-\beta^{j-i})}{(1-\beta)^2}+\beta^{j-i}Q_{jj}\\ &=\frac{\beta^2}{(1-\beta)^2}\left[n^2\left(\frac{1}{2}\right)-n\left(\frac{3}{2}+\frac{2\beta+\beta^n}{1-\beta}\right)\right.\\ &\left.+\left(1+\frac{2\beta^2-2\beta^n+\beta^3-\beta^{n+2}}{(1-\beta)^2}+\frac{4\beta+\beta^2}{1-\beta}+\beta^n\right)\right]\\ &+\left(\frac{1}{\beta-1}\right)\left(\frac{(\beta-1)(2\gamma-1+\beta)}{4(\gamma-1)(2\gamma-1-\beta)}\right)\\ &\times\left(\frac{(2\gamma-1)^2\beta}{2\gamma-1-\beta}\left[\frac{(2\gamma-1)-(2\gamma-1)^{n-1}}{2-2\gamma}-\frac{\beta-\beta^{n-1}}{1-\beta}\right]\right)\\ &-\left(\frac{2}{\beta-1}\right)\left(\frac{2\gamma-1-\gamma\beta}{(2\gamma-1-\beta)(1-\beta)}\right)\\ &\times\left(n\frac{\beta^3-\beta^{n+1}}{1-\beta}-\beta\left((n-1)\frac{\beta^2}{1-\beta}-\frac{\beta^3-\beta^{n+2}}{(1-\beta)^2}-\beta^n\right)\right)\\ &+\left(\frac{\beta+1}{2(\beta-1)}\right)\left(\frac{\beta}{1-\beta}\frac{n(n-1)-2}{2}+\frac{\beta}{(1-\beta)^2}(n-2)\right.\\ &\left.-\left(\frac{1}{(1-\beta)^2}+\frac{n}{1-\beta}\right)\frac{\beta^2-\beta^n}{1-\beta}\right)\\ &+\left(\frac{1}{\beta-1}\right)\left[\left(\frac{\beta+1}{4}\right)\frac{(2\gamma-1)}{\gamma-1}\right.\\ &\left.-\frac{\beta}{\beta-1}-\frac{(n+1)(\beta+1)}{2}\right]\frac{\beta}{1-\beta}\left(n-2-\frac{\beta-\beta^{n-1}}{1-\beta}\right)\end{aligned} \quad (1.7)$$

which accounts for all terms. Note that some algebraic simplification is possible but the equation has been left in the current form

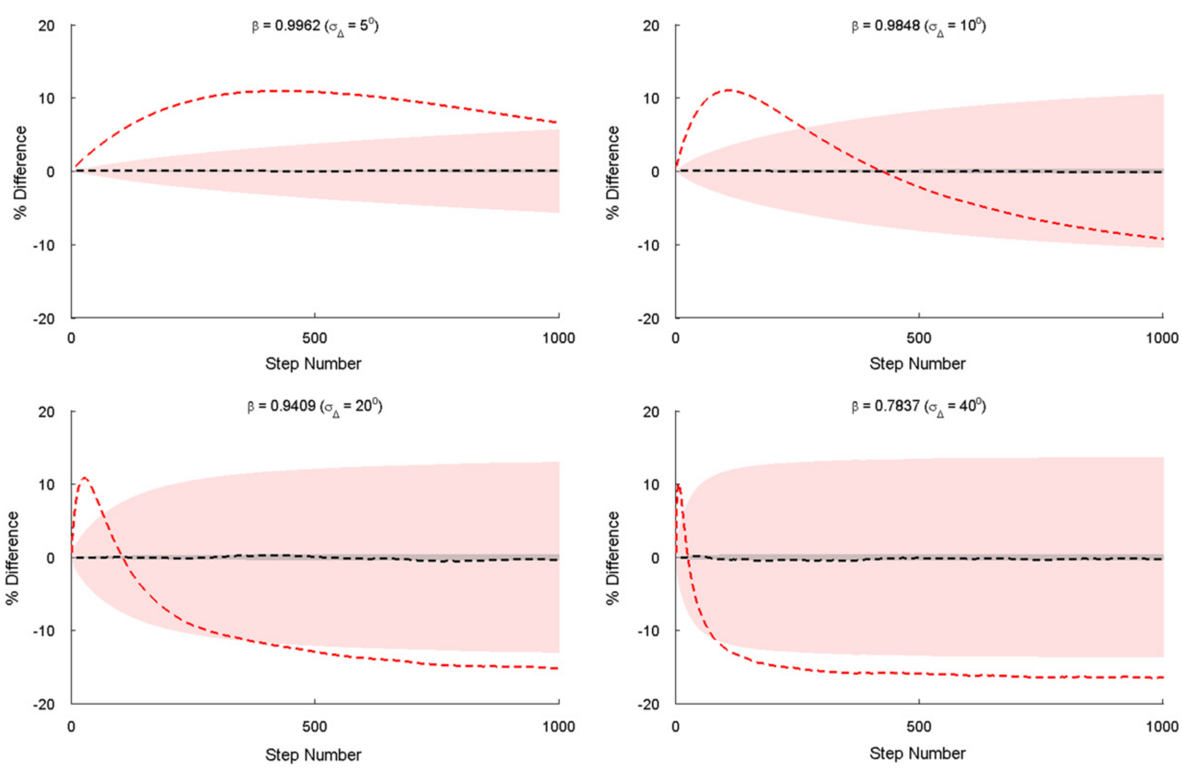

Fig. 1. The fourth radial moment of a discrete correlated/persistent random walk. The closed form solution reported by Claes and Van den Broeck (1987) (red dashed line, Eq. (1.2)), and the result derived above (black dashed line, Eq (1.3)–(1.7)) are expressed as a % difference from the Monte Carlo simulation results (10^6 particles). The 95% confidence interval for the sample mean, $\overline{R^4}$ (used as the Monte Carlo estimate of $\langle R^4\rangle$), from the simulation is shown for a particle population of 10^6 (gray region) and 10^3 (pink region), demonstrating the high intrinsic variability of the fourth moment of the radial displacement. In these four examples, the heading angle between successive unit steps varied randomly according to a Gaussian distribution i.e., $\Delta\sim N(0,\sigma_\Delta{}^2)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

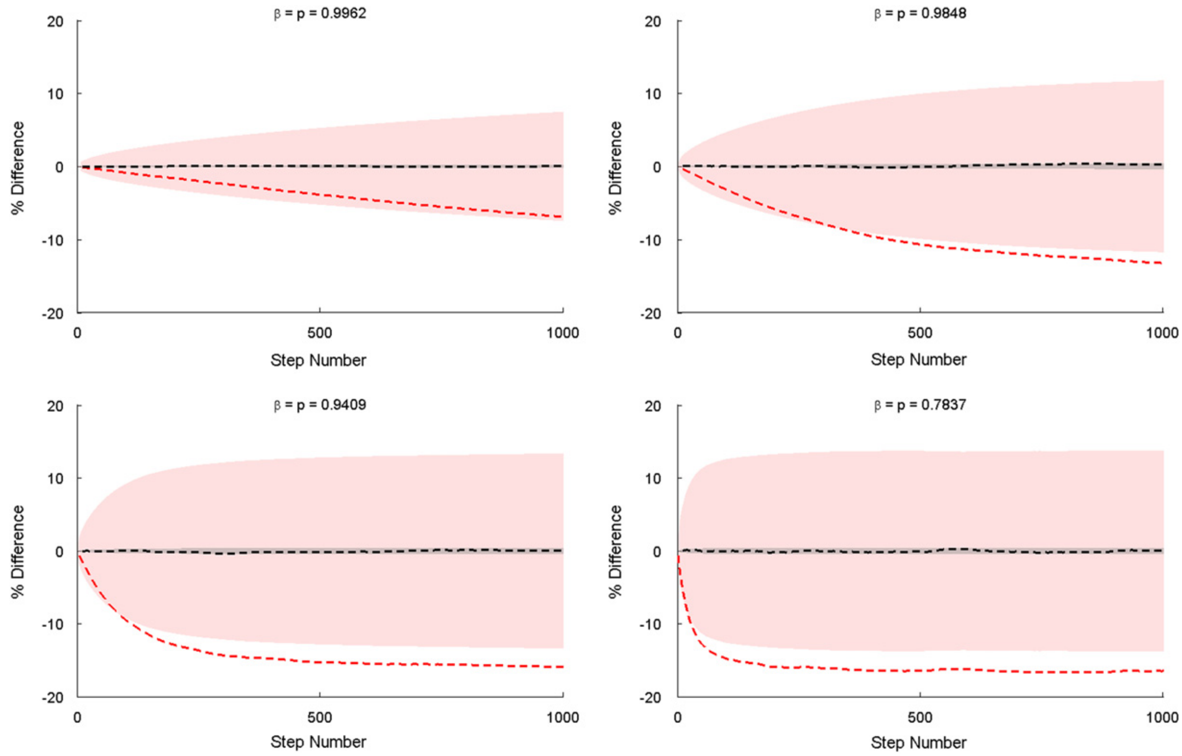

Fig. 2. A special case of the turn angle distribution. The closed form solution reported by Claes and Van den Broeck (1987) (red dashed line, Eq (1.2)), and the result derived above (black dashed line, Eq. (1.9)) are expressed as a % difference from the Monte Carlo simulation results (10^6 particles). In these four examples, the heading angle between successive unit steps varied randomly according to the polymer chain model described in Claes and Van den Broeck (1987). All other conventions are as per the previous figure. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

to remind the reader of the origins of the various contributions to the 4th moment. Since this sort of function will usually be evaluated by computer, the time cost has not been the concern of the current work. An important feature of this equation is the presence of $\gamma = \langle \cos^2 \Delta \rangle$ terms, which must exist in a general expression for $\langle R^4 \rangle$. The relationship between β and γ vary depending on the distribution of the turn angle Δ. For example, $\gamma = (1-\beta^4)/2$ for Gaussian distributions, $\gamma = (1+\beta^2)/2$ for wrapped Cauchy distributions, $\gamma = (1+\beta)/2$ for the Claes and Van den Broeck (1987) model's distribution, as well as many situations where the relationship is not expressible as a simple function. Therefore, it is impossible to write a general solution expressed purely in terms of either β or γ. Fig. 1 shows a comparison of Claes and Van den Broeck's result (1.2) and the earlier derived result (1.3–1.7) against Monte Carlo standard for Gaussian turn angles. It is worth noting that for small Monte Carlo particle clouds (e.g. 1000), the 95% confidence interval for the sample mean may be too large to detect small to moderate errors in a derived result.

On careful inspection, it is evident that there are terms in the general solution (1.3–1.7) which have denominators which may be zero if $2\gamma - 1 = \beta$. Interestingly, this occurs in the example of Claes and Van den Broeck (1987) i.e.,

$$\gamma = \langle \cos^2(\Delta) \rangle = p\cos^2(0) + \frac{(1-p)}{2\pi}\int_{-\pi}^{+\pi} \cos^2(\theta)d\theta = \frac{1+p}{2} \tag{1.8}$$

which seems a plausible explanation of why the form of their solution did not contain any $\gamma = \langle \cos^2(\Delta) \rangle$ terms. However, as seen in Fig. 2, and the special case of $\langle R^4 \rangle$ derived assuming $2\gamma - 1 = \beta$ (see Appendix B)

$$\langle R^4 \rangle = 2n^2 \frac{(1+\beta)^2 + 2\beta^{n+1}}{(1-\beta)^2} - n\frac{(1+\beta)^3 + 8\beta(1+\beta)}{(1-\beta)^3} + \frac{6\beta(1-\beta^n)(1+\beta)^2}{(1-\beta)^4} \tag{1.9}$$

which again differs from Eq. (15) of Claes and Van den Broeck (1987).

To examine this discrepancy further, it is possible to consider the asymptotic behaviour of $\langle R^m \rangle$. From (1.2–1.7) or 1.9, it is straightforward to show that for large n, the dominant terms are of $O(n^2)$. In fact, for $m=2$ and $m=4$, it can be shown that as $n \to \infty$

$$\lim_{n\to\infty} \langle R^m \rangle = \langle (X^2+Y^2)^{m/2} \rangle \approx n^{m/2}\Gamma(1+m/2)\left(\frac{1+\beta}{1-\beta}\right)^{m/2} \tag{1.10}$$

where $\Gamma(\cdot)$ is the gamma function. When $m=4$, $\Gamma(3)=2!=2$ but the numerical factor of the dominant term in Claes and Van den Broeck (1987) is 5/3. This appears to explain the underestimation for large n. Although empirically 1.10 seems to be a reasonable approximation for other (odd or even) positive integer values of m, it should be emphasized that this is merely an approximation, not an exact result. Although it is difficult to speculate on the source of the error in Eq. (15) of Claes and Van den Broeck (1987), it is now clear what the true solution is, where the dominant source of error resides algebraically, and how one might go about deriving similar results correctly for future applications. The derivation scheme used here may be generalized to obtain higher moments either manually, or via symbolic computational methods.

Correlated/persistent random walks have widespread applications in the biological and physical sciences. The radial moments are often used to quantify such random walks because there is no need to define an orientation *a priori*, which in fact may be unknown. The 4th radial moment $\langle R^4 \rangle$ is useful to obtain the exact variance of the squared radial dispersal, $V(R^2) = \langle R^4 \rangle - \langle R^2 \rangle^2$, of a correlated/persistent random walk. Furthermore, McCulloch and Cain (1989) estimated the first radial moment as $\langle R \rangle \approx \sqrt{\langle R^2 \rangle}\left\{1 - \frac{1}{8}V(R^2)/\langle R^2 \rangle^2\right\}$. Hence an exact $\langle R^4 \rangle$ has direct use in quantifying the variability of R^2 but also in estimating $\langle R \rangle$ and $V(R)$.

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Appendix A. Supplementary material

Supplementary data associated with this article can be found in the online version at doi:10.1016/j.jtbi.2010.02.038.

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